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# On Linearity in the Special Theory of Relativity

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## *Abstract*

In deductions of Lorentz transformations of the special theory of relativity, linearity of transformation is always postulated. There are only a few discussions about this linearity in which it is deduced from some basic physical facts. Here, it is shown to be almost a mathematical consequence of the principle of relativity.

## 1. *Introduction*

In order to deduce the law of transformation of coordinates, generally known as the Lorentz transformation, in the special theory of relativity, it is customary to assume that the law of transformation is a linear one.

Einstein, in his original paper (1905; compare also Bergmann, 1960) pointed out that the linearity of the transformation should be attributed to homogeneity of space and time. In other works (Einstein, 1955; compare also Pauli, 1921) linearity of transformations is considered as a consequence of the principle of constancy of velocity of light.

According to Pauli (1921) this linearity can be justified by the statement that a uniform rectilinear motion in one frame $K$ must also be rectilinear in the transformed frame $K'$. Further it is to be taken for granted that finite coordinates in $K$ remain finite in $K'$. Here $K$ and $K'$ refer to two reference systems in uniform rectilinear motion.

Fock discussed the question of linearity of transformation in some detail. In his general derivation of linearity, the starting fundamental postulates are:

'(i) To a uniform rectilinear motion in the coordinates $(x_i)$ there must exist a motion of the same nature in the transformed coordinates $(x_i')$.

(ii) To a uniform rectilinear motion with light velocity in the coordinates $(x_i)$ there corresponds a motion of the same nature in the coordinates $(x_i')$.

Here the index, $i$ runs through 0, 1, 2, 3.'

In the derivation of Fock, it is seen that from the postulate (i), the transformations of coordinates are obtained as those given by fractional linear functions with a common denominator (i.e. projective transformations of collineations), and from the postulate (ii) they are obtained as those given by rational functions, of which the numerators are linear and the common denominator is a quadratic of some special form (Möbius transformations), and from the two postulates taken together, linearity of transformation is obtained. Any one of the above two postulates, supplemented by the postulate that 'finite coordinates remain finite under transformations', implies the linearity of the transformations.

Thus, in the literature on the special theory of relativity, linearity is considered either as a simplifying assumption or as something connected with the special structure of the space-time continuum, like homogeneity of space and time, or with some basic physical facts, like existence of rectilinear motion, or constancy of the velocity of light along a line in vacuum. But from a close mathematical analysis it appears that linearity of transformations is neither an additional assumption nor a consequence of some other basic geometrical or physical facts, which in any strict mathematical development are considered as postulates.

The object of this paper is to show that linearity of transformations is implied by the simple fact of finite coordinates remaining finite under transformations, which is almost a necessary consequence of the principle of relativity. For this in Section 2, we shall discuss the mathematical nature of transformation laws of interest in physics, and clearly state the mathematical postulate. In elegant mathematical development of physical theories, it is essential to state clearly general mathematical postulates regarding functions used in the development, but, unfortunately, this cannot be seen in the discussions of the special theory of relativity. Then in Section 3 linearity of transformation will be deduced from the simple fact that finite coordinates remain finite. In Section 4, the nature of implication of the said fact of finite coordinates remaining finite by the principle of relativity is discussed. In concluding remarks, the reason why global linearity does not appear in the general theory of relativity is briefly pointed out.

## 2. *Mathematical Nature of Transformation Laws*

In physics, one is interested to have the laws expressed in terms of linear functions or polynomial functions, or, in complicated cases, of rational, or algebraic, or some well-known transcendental functions. In geometry (science of space) transformations are 'nothing more than a generalisation of function' (Klein, 1925). For application in physics in general, and in the special theory of relativity in particular, the point transformations of geometries, in which the points are base elements and points are transformed into points, are useful. The analytic expression of a point transformation in $n$-dimensional space is given by

$$x_i' = f_i(x_1, x_2, \ldots, x_n) \qquad i = 1, 2, \ldots, n$$

where $(x_1, x_2, \ldots, x_n)$ and $(x_1', x_2', \ldots, x_n')$ are coordinates of a point with respect to frames of reference $K$ and $K'$. In general theories of transformations, $f_i$'s, the correspondence defined as above is reversibly one-valued and continuous. But in quantitative discussions, a specification as linear function or rational functions or the like is necessary.

Following Einstein (1905), the space-time continuum $[R^4]$ is four-dimensional, i.e. a point is given by a quadruple $\{x_i\}$, where $i$ can have values 1, 2, 3, 4. In some developments, $i$ is taken to have values 0, 1, 2, 3. Here the range of $i$ will be taken as 1, 2, 3, 4. In the first paper of Einstein (1905), in the beginning, for the introduction of simultaneity, time (proportional to the fourth coordinate) is taken as a general analytic function of coordinates and then it is approximated to by a linear function when the space coordinates are chosen 'infinitesimally small', i.e. he had Taylor expansion of the function (at least in finite form), in his mind.

In his book (Einstein, 1955), in the general discussion, of coordinate transformations of pre-relativistic physics, Einstein took $f_i$'s, where $i = 1, 2, 3$, as functions expressible in Taylor series, and subsequently he used this discussion in the background. Fock (1959) took $f_i$'s as functions having continuous partial derivatives of second order.

In Weierstrassian development of function theory, functions are introduced by power series. All the functions of frequent use in general quantitative discussion of mathematics and physics are expressible in power series in some domains, in which they are called analytic. When $f_i$'s are sought in the form with a linear part, together with terms involving higher power of $x_i$'s, $f_i$'s are really taken as analytic, at least locally. Then the fact that finite coordinates remain finite in transformations implies that $f_i$'s are bounded. Thus the transformation functions are bounded in any finite region of $R^4$, and are analytic everywhere except at singular points. Now, if $R^4$ be embedded in the usual way (Synge, 1956) in $C^4$ by replacing $x_j$'s by $z_j$'s, where $z_j = x_j + iz_j$; $j = 1, 2, 3, 4$ then the transformation functions can easily be extended analytically from any domain $D_r$ of $R^4$ to a domain $D_c$ of $C^4$. It is well known in the theory of complex functions of several variables (Fuks, 1963) that *if a function $f(z_1, z_2, z_3, z_4)$ be analytic in the entire closed polycylinder, $D \equiv \{|z_k - z_0| \leqslant \gamma_k,\ k = 1, 2, 3, 4\}$ with possible exclusion of some set $E \subset D$ about which to each point with first three coordines $(\alpha_1, \alpha_2, \alpha_3)$ of the closed polycylinder $D_j = \{|z_k - z_0| \leqslant \gamma_k,$ $k \neq j$ and $k, j = 1, 2, 3, 4$, for a fixed $j$ there corresponds at most a finite set* $(\alpha_1, \alpha_2, \alpha_3, \beta_s)$, $s = 1, 2, 3, 4, \ldots$, lying in $E$, while $|\beta_s| < \gamma_4$, and if $f(z_1, z_2, z_3, z_4)$ *be bounded in each region $D_0$, where $\bar{D}_0 \subset D$, then its values at the points of* *the set E may be revised or extended so that the function may be analytic in $\bar{D}$.* The above theorem is an alternative generalised version of the theorem of removable singularity of Riemann. Moreover, in consistency with the general principle of continuity of macroscopic physics, the transformation function should be assumed to be continuous. It is also known, from the theory of functions of several complex variables (Fuks, 1963), that *if a function $f(z_1,z_2,z_3,z_4)$ be continuous in a neighbourhood U of a point P in $C^4$ and analytic in U, except possibly on points on a hypersurface $\Sigma$ in $C^4$ containing P as its ordinary point and given by*

$$y_4 = \phi(x_1, x_2, x_3, x_4, y_1, y_2, y_3) \tag{2.1}$$

*then the function may be made analytic in U.* Unless the singular points of the transformations are distributed most unfavourably they can be removed and the transformation functions can be taken to be analytic in same domain in $C^4$ containing $R^4$ and so in $R^4$. So, without loss of any generality it is possible to assume that the transformation, besides being one-to-one, is represented by functions which are analytic in the entire $R^4$.

## 3. *Linearity of Transformation Law*

In the preceding section, it is seen that the transformation functions are analytic in $R^4$. They are expressible as a power series about origin as

$$\begin{aligned} f_i(x_1,x_2,x_3,x_4) &= \sum_{j,k,l,m} a^{(i)}_{jklm} x_1{}^j x_2{}^k x_3{}^l x_4{}^m \\ i = 1,2,3,4 \quad & j,k,l,m = 0,1,2,\ldots \end{aligned} \tag{3.1}$$

Then, the complex valued functions of complex variables,

$$f_i(z_1,z_2,z_3,z_4) = \sum_{j,k,l,m} a^{(i)}_{jklm} z_1{}^j z_2{}^k z_3{}^l z_4{}^m \qquad i = 1,2,3,4 \tag{3.2}$$

which are evidently a natural analytic continuation of the real functions given by (3.1) in $R^4$, cannot have finite associated radii of convergence (Fuks, 1963) and hence are integral functions. The uniqueness of analytic continuation are guaranteed by the following theorem of mathematical analysis (Dieudonné, 1964).

> *'Let $A \subset C^p$ be an open connected set, f and g two analytic functions in A with values in a complex Banach space F. Let U be an open subset of A, b a point in U and suppose that $f(\mathbf{x}) = g(\mathbf{x})$ in the set $U \cap (b + R^p)$ then $f(\mathbf{x}) = g(\mathbf{x})$ for every $x \in A$.'*

As $C^4$ is itself a complex Banach space and $A$ and $U$ may be taken as $C^4$ and $b$ as the origin, the result is evident.

*Proposition:* Transformation functions $f_i$ are linear.

*Proof:* Now, the series, $\sum a^{(i)}_{pqrs} z_1{}^p z_2{}^q z_3{}^r z_4{}^s$, for $f_i$, $i = 1,2,3,4$ are absolutely convergent, and so, unconditionally convergent in the entire (open) $C^4$.

So, by rearrangements of terms, $f_i$'s can be written in the form of a psuedo-polynomial as:

$$f_i(z_1, z_2, z_3, z_4) = \sum_{n=0}^{\infty} A_n^{(i)}(z_j, z_k, z_l) z_m^n, \qquad (3.3)$$
$$i, j, k, l, m = 1, 2, 3, 4 \qquad j \neq k \neq l \neq i$$

where $A_n^{(i)}(z_j, z_k, z_l)$ are analytic functions of $z_j$, $z_k$, $z_l$. As $f_i$'s transform points at infinity only to points at infinity, so, for each fixed set of finite values of $z_j$, $z_k$, $z_l$ the series in the right-hand side reduces to a polynomial of one variable, $z_m$. As $f_i$'s are one-to-one for each set of $(z_1, z_2, z_3, z_4)$, so the pseudo-polynomial must be linear in $z_m$ $(m = 1, 2, 3, 4)$. Thus

$$f_i(z_1, z_2, z_3, z_4) = \sum_{j=1}^{4} a_j^{(i)} z_j, \qquad i = 1, 2, 3, 4 \qquad (3.4)$$

*Remarks*

Now, an alternative sophisticated proof can also be sketched by using two known theorems (Fuks, 1963) in the following way. For discussion with points at infinity, it is seen to be convenient to extend the usual space $C_4$ to a space $P_4$ by the compactification of the image of the space $C_4$ in the space $P_4$ by points at infinity, where $P_4$, in distinction from $C_4$, is compact. It is a well-known practice to interpret the usual coordinate transformations of a space as its mapping to itself (Synge, 1956). Then one can use the theorem: 'The most general meromorphic (in the strict sense) mapping of the whole projectively completed space $P_4$ into itself has the form

$$z_j{}^1 = \frac{a_0^{(i)} + \sum_{j=1}^{4} a_j^{(i)} z_j}{b_0^{(i)} + \sum_{j=1}^{4} b_j^{(i)} z_j} \qquad (3.5)$$

that is, it is a projective transformation' (Fuks, 1968) where a continuous mapping $T$ of the region $D \subset P_4$ is said to be meromorphic (in the strict sense) if it is analytic at all points $z$ of $D$, for which $T(z) \in T(D) \subset C^4$, and at the points $z$ of $D$ to which there correspond points at infinity, the mapping $\pi T$ is analytic, where $\pi$ is an appropriate projective mapping which carries the point $T(z)$ into finite $z'$.

Now,

$$Q = \frac{1}{b_0^{(i)} + \sum_{j=1}^{4} b_j^{(i)} z_j} \qquad (3.6)$$

is zero at points at infinity. As for finite $z_j$'s, $z_j'$ are finite, so, $|Q|$ is finite for finite $z_j$'s. Thus, by Liouville's theorem $Q$ is a constant, i.e. the transformations (3.6) are linear.

## 4. *The Principle of Relativity and Linearity*

As already discussed in Section 2, the main aim of the majority of physical investigations is to express a law of nature in the form $F(\mathbf{x}) = 0$, where $F(\mathbf{x})$ is locally analytic function in $R_4$ in terms of a reference system $K$. With reference to another reference system $K'$ a law of nature is to be obtained as

$$F'(\mathbf{x}') \equiv F(f(\mathbf{x}')) = 0$$

There are various equivalent formulation of the principle of relativity in the literature. In the language of Fock (1959): as

> '*... the principle of relativity asserts that the two sequences of events will be exactly the same (at least in so far as they are determined at all.) If a process in the original system can be described in terms of certain functions of the space and time coordinates of the first frame, the same functions of space and time coordinates of the second frame will describe a process occurring in the copy.*'

Here, a process refers to a physical process or event and frames, to frames of reference $K$ and $K'$.

In the strict sense, that $F$ and $F'$ are same, means that they are of same forms in terms of variables, or in mathematical terminology, $F$ is an automorphic function with respect to group of transformations (i.e. admissible coordinate transformations). In the theory of automorphic functions (Ford, 1951; Dutta & Debnath, 1965), and also in the investigations of invariant functions under a group of coordinates transformations, the group of transformations are given and the form of the functions automorphic with respect to the group is found out. But, in the special theory of relativity the first problem is to find out the group of transformations with respect to which the function expressing physical process are automorphic. As up to now the physical investigations are mostly based on mechanics or the theory of electromagnetism, in the special theory, basic physical processes are taken as laws of mechanics and/or the constancy of velocity of light (or invariance of the wave front). But if basic physical facts are not introduced and the nature of physical processes is kept open, then the main problem does not appear to be solvable in its generality, but the linearity of coordinates transformation can be inferred. For this, by the word 'same' in the formulation, we need not mean 'of the same form'. If the term 'same' is taken as to mean 'of the same function-theoretic nature' will be sufficient. If $F$ and $F'$ have always the same number of singularities of the same nature (where multiplicity has been taken into account), $F$ and $F'$ are considered as 'same' function in function-theoretic discussions. The latter sense of the term 'same' is much weaker than that of being automorphic. Then for function-theoretic discussions, $f$'s defined in domains, $D_R$ of $R_4$ may be extended to domains $D_C$ of $C^4$ when $D_C \supset D_R$ and considered as functions of $z$, $z \in D_C$. In order that $F$ and $F'$ are same always, it is sufficient that the transformation functions $z = \phi(z')$ must not have any singularity for every $D_C$ of analyticity of $F$'s. Unless all $F$'s expressing different physical laws be of such form that the principal parts at every singularity of $\phi$'s (not of $F$'s) cancel each other (an assumption which is highly unlikely), the transformation laws are necessarily analytic. Thus $\phi$'s must be analytic and so bounded in every finite domain. Thus it remains valid in every finite domain $D_R \subset D_C$. This leads to the fact that finite coordinates in $K$ remain finite in $K'$.

## *Concluding Remarks*

Here, discussions are made generally from mathematical analysis, from the principle of relativity and the general observations of mathematical nature of physical laws without the introduction of the specific physical, mechanical or geometrical postulates. Then, naturally the question why the linearity is not significant in the general theory of relativity becomes important. The answer to this lies in the fact that the present discussion is based on the existence of coordinate systems in which each coordinate varies from $-\infty$ to $+\infty$ along a real line. When in a theory a broader class of coordinate systems are included, the above discussions are not valid. For this, linearity is not to be used in the general theory of relativity.

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